Derivation:
Transforms of the Wave Equation

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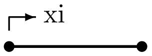
1 Fixed End Conditions

The wave equation, when normalized (but not yet made dimensionless to eliminate c) has the basic form

$$\left[\frac{\partial^2 u}{\partial t^2}\right]_{\xi=\text{const}} - c^2\left[\frac{\partial^2 u}{\partial \xi^2}\right]_{t=\text{const}} = 0 \tag{1}$$

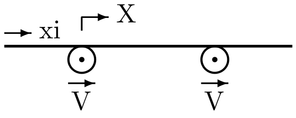


where u can be a lateral deflection of a string, rope, cable, or chain; or longitudinal deflection of a rod or cable; or rotational deflection of a shaft. Correspondingly, $c^2 = (T/\rho A)$, or $(EA/\rho A)$, or $(GJ/\rho J)$, respectively.[1] The independent variable ξ measures location relative to a point on the string/rod/shaft; the string/rod/shaft must be either stationary, or else moving at a constant velocity and direction, so that the attached coordinate system is inertial.

The equation can be expanded to include slender tensioned beams (and compressed columns) by adding the bending-stiffness term $+(EI/\rho A)\left[\partial^4 u/\partial \xi^4\right]$, which makes the equation fourth-order.

2 Threadlines

If the string/rod/shaft translates at a constant velocity $-V$ relative to the supports, or else both end conditions uniformly translate at a velocity V relative to the string/rod/shaft,



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[1] *vide* Peter M. Moretti, *Modern Vibrations Primer,* CRC Press, Boca Raton, ©2000 (ISBN 0-8493-2038-0), pages 292, 305, & 307.

the wave equation still applies, but it is convenient to transform to a coordinate x defined by

$$\begin{aligned} x + Vt &= \xi \\ x &\doteq \xi - Vt \end{aligned}$$

in order to make the end conditions stationary.[2] Transforming the wave equation

$$\left[\frac{\partial^2 u}{\partial t^2}\right]_{(x+Vt)} - c^2 \left[\frac{\partial^2 u}{\partial (x+Vt)^2}\right]_t = 0$$

we examine and expand each term

$$\begin{aligned} \left[\frac{\partial u}{\partial t}\right]_{(x+Vt)} &= \left[\frac{\partial u}{\partial t}\right]_x - V\left[\frac{\partial u}{\partial x}\right]_t \\ \left[\frac{\partial^2 u}{\partial t^2}\right]_{(x+Vt)} &= \left[\frac{\partial^2 u}{\partial t^2}\right]_x - V\left[\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial x}\right]_t\right]_x - V\left[\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial t}\right]_x\right]_t + V^2\left[\frac{\partial^2 u}{\partial x^2}\right]_t \\ \left[\frac{\partial^2 u}{\partial (x+Vt)^2}\right]_t &= \left[\frac{\partial^2 u}{\partial x^2}\right]_t \text{ and, if needed, } \left[\frac{\partial^4 u}{\partial (x+Vt)^4}\right]_t = \left[\frac{\partial^4 u}{\partial x^4}\right]_t \end{aligned}$$

to obtain the governing equation

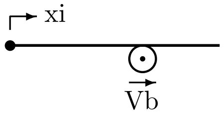
$$\left[\frac{\partial^2 u}{\partial t^2}\right]_{x=\text{const}} - 2V\left[\frac{\partial^2 u}{\partial t \partial x}\right] + \left(V^2 - c^2\right)\left[\frac{\partial^2 u}{\partial x^2}\right] = 0 \tag{2}$$

Reversing the coordinate directions changes only the sign of the mixed-derivative term. Stability requires that

$$V^2 \le c^2$$

3 One-sided case

If a stationary string/rod/shaft has a stationary end condition at $x = \xi = 0$, and the other end condition translates at a constant velocity V_b relative to the string, so that the length changes from an initial value of $L_o > 0$ to $L_{(t)} = L_o + V_b t$,



it is convenient to transform ξ to a coordinate

$$\begin{aligned} z &\doteq \xi - \frac{\xi}{L_o + V_b t} V_b t = \frac{\xi L_o}{L_o + V_b t} \\ \xi &= \frac{L_o + V_b t}{L_o} z = z + \frac{V_b t z}{L_o} \end{aligned}$$

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[2] *ibidem*, page 302.

so that $z = 0$ at the fixed left end, and $z = L_o$ at $\xi = L_o + V_b t$, which makes both end conditions stationary. Transforming the wave equation

$$\left[\frac{\partial^2 u}{\partial t^2}\right]_{z+V_b tz/L_o} - c^2 \left[\frac{\partial^2 u}{\partial \left(z + V_b tz/L_o\right)^2}\right]_t = 0$$

we examine and expand each term

$$\begin{aligned}
\left[\frac{\partial u}{\partial t}\right]_{z+V_b tz/L_o} &= \left[\frac{\partial u}{\partial t}\right]_{z=\text{const}} - \frac{V_b z}{L_o}\left[\frac{\partial u}{\partial z}\right]_t \\
\left[\frac{\partial^2 u}{\partial t^2}\right]_{z+V_b tz/L_o} &= \left[\frac{\partial^2 u}{\partial t^2}\right]_z - \frac{V_b z}{L_o}\left[\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial x}\right]_t\right]_z - \frac{V_b z}{L_o}\left[\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial t}\right]_z\right]_t + \left(\frac{V_b z}{L_o}\right)^2\left[\frac{\partial^2 u}{\partial z^2}\right]_t
\end{aligned}$$

$$\begin{aligned}
\left[\frac{\partial^2 u}{\partial \left(z + V_b tz/L_o\right)^2}\right]_t &= \frac{1}{\left(1 + \frac{V_b t}{L_o}\right)^2}\left[\frac{\partial^2 u}{\partial z^2}\right]_t \\
\left[\frac{\partial^4 u}{\partial \left(z + V_b tz/L_o\right)^4}\right]_t &= \frac{1}{\left(1 + \frac{V_b t}{L_o}\right)^4}\left[\frac{\partial^4 u}{\partial z^4}\right]_t
\end{aligned}$$

and obtain the governing equation

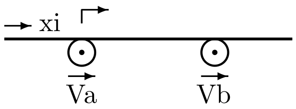
$$\left[\frac{\partial^2 u}{\partial t^2}\right]_z - 2\frac{V_b z}{L_o}\frac{\partial^2 u}{\partial t \partial z} + \left(\left(\frac{V_b z}{L_o}\right)^2 - \left(\frac{cL_o}{(L_o + V_b t)}\right)^2\right)\frac{\partial^2 u}{\partial z^2} = 0 \tag{3}$$

which is linear, but with non-constant coefficients. To avoid a divergent solution in the domain $0 < z < L_o$, we specify

$$\begin{aligned}
V_b^2 &\leq \left(\frac{cL_o}{(L_o + V_b t)}\right)^2 \\
V_b^4\left(\frac{t}{L_o}\right)^2 + 2V_b^3\left(\frac{t}{L_o}\right) + V_b^2 &\leq c^2 \\
\frac{V_{crit} t}{L_o} &= \frac{1}{2}\left(\pm\sqrt{1 \pm 4\frac{ct}{L_o}} - 1\right) \\
\text{for positive } V_b,\ \frac{V_{crit} t}{L_o} &= \frac{1}{2}\left(\sqrt{1 + 4\frac{ct}{L_o}} - 1\right) \approx \frac{ct}{L_o} \\
&\Longrightarrow V_{crit} \approx c \\
\text{for negative } V_b,\ \frac{V_{crit} t}{L_o} &= \frac{1}{2}\left(-\sqrt{1 + 4\frac{ct}{L_o}} - 1\right) \approx -1 \\
&\Longrightarrow t_{\max} \approx -L_o/V_b
\end{aligned}$$

4 Unsymmetrical case

If one end condition translate at a constant velocity V_a relative to the string, and the other end condition translates at a constant velocity V_b relative to the string, so that the length changes from an initial value of L_o to $L_{(t)} = L_o - V_a t + V_b t$,



it is convenient to transform to a coordinate

$$\begin{aligned}
z &\doteq (\xi - V_a t) - \frac{(\xi - V_a t)}{L_o + (V_b - V_a)\,t}\,(V_b - V_a)\,t \\
&\boxed{\tfrac{z}{L_o} = \tfrac{\xi - V_a t}{L_o + (V_b - V_a)t}} \\
\xi &= \frac{L_o + (V_b - V_a)\,t}{L_o} z + V_a t \\
&= z + \frac{V_b t z}{L_o} - \frac{V_a t z}{L_o} + V_a t
\end{aligned}$$

so that $z = 0$ at the left end where $\xi = V_a t$, and $z = L_o$ at $\xi = L_o + V_b t$, which makes both end conditions stationary. Check that this reduces to the previous three cases

$$\begin{aligned}
\text{when } V_a &= 0 \\
\text{then } z &= \xi - \frac{\xi}{L_o + V_b t} V_b t \\
\text{and } \xi &= \frac{L_o + V_b t}{L_o} z \\
\text{or when } V_a &= V_b = V \\
\text{then } z &= \xi - Vt \\
\text{and } \xi &= z + Vt \\
\text{or when } V_a &= V_b = 0 \\
\text{then } z &= \xi
\end{aligned}$$

Transforming the wave equation

$$\left[\frac{\partial^2 u}{\partial t^2}\right]_{z + V_b t z/L_o - V_a t z/L_o + V_a t} - c^2 \left[\frac{\partial^2 u}{\partial\left(z + V_b t z/L_o - V_a t z/L_o + V_a t\right)^2}\right]_t = 0$$

we examine and expand each term

$$\begin{aligned}
\left[\frac{\partial u}{\partial t}\right]_{z+V_btz/L_o-V_atz/L_o+V_at} &= \left[\frac{\partial u}{\partial t}\right]_{z=\text{const}} - \left(\frac{V_bz}{L_o}-\frac{V_az}{L_o}+V_a\right)\left[\frac{\partial u}{\partial z}\right]_t \\
\left[\frac{\partial^2 u}{\partial t^2}\right]_{z+V_btz/L_o-V_atz/L_o+V_at} &= \left[\frac{\partial^2 u}{\partial t^2}\right]_z - 2\left(\frac{V_bz}{L_o}-\frac{V_az}{L_o}+V_a\right)\left[\left[\frac{\partial^2 u}{\partial t\partial x}\right]\right] \\
&\quad + \left(\frac{V_bz}{L_o}-\frac{V_az}{L_o}+V_a\right)^2\left[\frac{\partial^2 u}{\partial z^2}\right]
\end{aligned}$$

$$\begin{aligned}
\left[\frac{\partial^2 u}{\partial\left(z+V_btz/L_o-V_atz/L_o+V_at\right)^2}\right]_t &= \frac{1}{\left(1+\frac{V_bt}{L_o}-\frac{V_at}{L_o}\right)^2}\left[\frac{\partial^2 u}{\partial z^2}\right]_t \\
\left[\frac{\partial^4 u}{\partial\left(z+V_btz/L_o-V_atz/L_o+V_at\right)^4}\right]_t &= \frac{1}{\left(1+\frac{V_bt}{L_o}-\frac{V_at}{L_o}\right)^4}\left[\frac{\partial^4 u}{\partial z^4}\right]_t
\end{aligned}$$

and obtain the governing equation

$$\left\{\begin{array}{c}
\left[\frac{\partial^2 u}{\partial t^2}\right]_z - 2\left(\frac{V_bz}{L_o}-\frac{V_az}{L_o}+V_a\right)\frac{\partial^2 u}{\partial t\partial z} \\
+\left(\left(\frac{V_bz}{L_o}-\frac{V_az}{L_o}+V_a\right)^2-\left(\frac{cL_o}{L_o+(V_b-V_a)t}\right)^2\right)\frac{\partial^2 u}{\partial z^2}
\end{array}\right\}=0 \tag{4}$$

Check: when $V_a = 0$ (lengthening to the right if V_b is positive, shortening if negative)

$$\left[\frac{\partial^2 u}{\partial t^2}\right]_z - 2\left(\frac{V_bz}{L_o}\right)\frac{\partial^2 u}{\partial t\partial z} + \left(\left(\frac{V_bz}{L_o}\right)^2-\left(\frac{cL_o}{(L_o+V_bt)}\right)^2\right)\frac{\partial^2 u}{\partial z^2}=0 \tag{3}$$

or when $V_b = 0$ (shortening from the left if V_a is positive, lengthening if negative)

$$\left[\frac{\partial^2 u}{\partial t^2}\right]_z - 2\left(V_a-\frac{V_az}{L_o}\right)\frac{\partial^2 u}{\partial t\partial z} + \left(\left(V_a-\frac{V_az}{L_o}\right)^2-\left(\frac{cL_o}{L_o-V_at}\right)^2\right)\frac{\partial^2 u}{\partial z^2}=0$$

or when $V_a = V_b = \pm V$

$$\left[\frac{\partial^2 u}{\partial t^2}\right]_z \pm 2V\frac{\partial^2 u}{\partial t\partial z} + \left(V^2-c^2\right)\frac{\partial^2 u}{\partial z^2}=0 \tag{2}$$

or when $V_a = V_b = 0$

$$\left[\frac{\partial^2 u}{\partial t^2}\right]_z + c^2\frac{\partial^2 u}{\partial z^2}=0 \tag{1}$$

5 Application

For the “Spaghetti” problem, Equation 3 can be interpreted as a string uniformly moving to the right at velocity $V_b = -V$ (with the inertial coordinate system moving with it), retracting into a support at the left. If we let the right end be a free end, making $T = 0$, and permitting some beam stiffness, the governing equation is

$$\left[\frac{\partial^2 u}{\partial t^2}\right]_z + 2\left(\frac{Vz}{L_o}\right)\frac{\partial^2 u}{\partial t \partial z} + \left(\frac{Vz}{L_o}\right)^2 \frac{\partial^2 u}{\partial z^2} + \left(\frac{EI}{\rho A}\right)\frac{\partial^4 u}{\partial z^4} = 0$$

with the solution going singular when $z \Rightarrow 0$. Conversely, by changing the sign on the mixed derivative, this equation covers uniform extrusion, observing from a starting length of $L_o > 0$.

Addendum

For comparison with the transformation approach above, we can also derive the Equation of Motion of a Threadline from Lagrange’s Equation:[3]

The Kinetic Energy $\mathfrak{T}$ for a string of density ρ, constant cross-sectional area A, length L, and traveling with a transport speed V, is, using the coordinates u and x of Section 2, dots for $\partial/\partial t$, and primes for $\partial/\partial x$,

$$\mathfrak{T} = \frac{1}{2}\int_0^L \rho A\left[\left(\dot{u} + Vu'\right) + V^2\right] dx$$

For constant tension T in the string, the Potential Energy $\mathfrak{U}$ is

$$\mathfrak{U} = \frac{1}{2}\int_0^L T\left(u'\right)^2 dx$$

Using Hamilton’s Principle $\delta \int_{t_t}^{t_2} \mathfrak{L} dt = 0$, where $\mathfrak{L} = \mathfrak{T} - \mathfrak{U}$, yields

$$\int_{t_t}^{t_2}\int_0^L \left[\rho A\left(\dot{u} + Vu'\right)\left(\delta\dot{u} + V\delta u'\right) - Tu'\delta u'\right] dx dt = 0$$

or

$$\left\{\begin{array}{c} \int_{t_t}^{t_2}\left[\rho AV\left(\dot{u} + Vu'\right) - Tu'\right]\delta u\,|_0^L\, dt \\ -\int_{t_t}^{t_2}\int_0^L\left[\rho A\left(\ddot{u} + 2V\dot{u}' + V^2 u''\right) - Tu''\right]\delta u dx dt \end{array}\right\} = 0$$

Therefore, the Equation of Motion is

$$\rho A\left(\ddot{u} + 2V\dot{u}' + V^2 u''\right) - Tu'' = 0$$

$$\text{or, } \ddot{u} + 2V\dot{u}' + \left(V^2 - c^2\right)u'' = 0$$

This is the same result as Equation 2 above; $c^2 = T/\rho A$ is the wave speed, and the End Conditions are $u_{(0,t)} = 0 = u_{(L,t)}$.

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[3] contributed July 2003 by Prof. Anirvan Dasgupta, Dept. of Mechanical Engineering, IIT Kharagpur – 721 302, India.