

**FLAG FLUTTER AMPLITUDES**

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*Over a substantial range of experimental parameters, flag flutter consists of traveling waves growing in amplitude as they progress towards the trailing edge. The out-of-plane motion of the trailing edge follows a curved path, and tension is dynamically induced by the centrifugal force. For a given flutter mode, dynamically induced tension can be obtained as a function of time and location; time-averaged tension depends only on the square of the velocity amplitude and the mass of the oscillating fabric. The tension opposes the flutter deflections induced by the flow field. It is hypothesized that dynamically induced tension is the main non-linear term limiting post-critical amplitudes of flexible flags, and a major factor in cantilevered panel flutter and in edge flutter of paper and plastic webs. To examine this hypothesis, published experimental observations of flag flutter are reviewed.*

**1 Introduction**

Flag flutter is a challenging problem in physics. The stability limit is very low—flags can be observed to flutter even at very low wind speeds—and the flutter amplitudes and modes are steady. The prediction of flag-flutter amplitudes is a fundamental problem in the study of fluid/structure interaction.

In an earlier paper (Moretti, 2003) we presented the governing equation for the flutter motion of cantilevered panels and flags

$$\begin{aligned}
& m \frac{\partial^2 w}{\partial t^2} \\
& - \left\{ T \left[ 1 + \frac{3}{2} \left( \frac{\partial w}{\partial s} \right)^2 \right] + m \int_s^L \int_0^s \left( \left( \frac{\partial^2 w}{\partial t \partial s} \right)^2 + \frac{\partial w}{\partial s} \frac{\partial^3 w}{\partial t^2 \partial s} \right) ds ds \right\} \frac{\partial^2 w}{\partial s^2} \\
& - \left\{ \frac{\partial T}{\partial s} \left[ 1 + \frac{1}{2} \left( \frac{\partial w}{\partial s} \right)^2 \right] + m \int_0^s \left( \left( \frac{\partial^2 w}{\partial t \partial s} \right)^2 + \frac{\partial w}{\partial s} \frac{\partial^3 w}{\partial t^2 \partial s} \right) ds \right\} \frac{\partial w}{\partial s} \\
& + EI \left\{ \frac{\partial^4 w}{\partial s^4} \left[ 1 + \left( \frac{\partial w}{\partial s} \right)^2 \right] + 4 \frac{\partial w}{\partial s} \frac{\partial^2 w}{\partial s^2} \frac{\partial^3 w}{\partial s^3} + \left( \frac{\partial^2 w}{\partial s^2} \right)^3 \right\} \\
& = \Delta p
\end{aligned} \tag{1}$$

where  $w$  is the out-of-plane deflection as a function of time  $t$  and location  $s$  measured along the flag, with mass-per-unit-area  $m$ , stiffness-per-unit-width  $EI$ , friction-drag-induced tension-per-unit-width  $T$ , and excitation by pressure-difference across the flag  $\Delta p$ .

We note that there is a tension-like double-integral term, and a tension-derivative-like integral term; we designate them “dynamically induced tension.” When deriving the partial differential equation from Hamilton’s principle, these terms come from the in-plane kinetic energy in Lagrange’s equation, and are inherently non-linear.

To judge the importance of these terms, we should observe the motion, evaluate all terms, and compare their average magnitudes.

## 2 Structural Model

For studying dynamically induced tension, flags offer analytical simplicity:

1. Flags are *flexible*: the plate-stiffness term is small relative to other terms.
2. Flags are *inextensible*: the path length from luff to leech is nearly constant; the curvature is cylindrical or conical;
3. if the orientation is horizontal, the deflections are uniform across the width.
4. Flags can be made *wide* compared to the wave-length, so that the flow-field is two-dimensional;
5. for pressure calculations, the far-field flow approximates potential flow.

When a flag flutters under the influence of the fluctuating pressure forces of the complementary flow field, the trailing edge travels a curved path and develops centrifugal forces which stretch the flag. This dynamically induced tension is felt at the attachment point as increased drag, over and above the drag due to boundary-layer friction. To calculate this tension, we must know the wave motion.

As an approximation, we use a simple description of the deflection  $w$  in the  $z$ -direction (normal to the initial plane of the flag) which captures the essential features of the observed motion: a *traveling wave* within an envelope growing from a value of zero at the leading edge to the maximum amplitude at the trailing edge. The simplest case is a *linearly increasing envelope* for the traveling wave

$$w = \frac{Ax}{L} \cos(\omega t - \kappa x) \quad (2)$$

$$\frac{\partial w}{\partial t} = \frac{-A}{L} x \omega \sin(\omega t - \kappa x) \quad (3)$$

where  $A$  is the maximum amplitude;  $x$  is the distance along the initial plane of the flag, measured from the support;  $L$  is the maximum value of  $x$ ; and the frequency of the waves is  $f = \omega/2\pi$ . In this first approximation,  $\kappa$  is taken to be a constant; therefore the *wave-length is constant* at  $\lambda = 2\pi/\kappa$ ; and the phase velocity is also constant at  $c = f\lambda = \omega/\kappa$ .

Thoma (1939) showed that the *time-averaged* tension dynamically induced by normal forces depends on the average of the square of the velocity

$$\frac{d[T]_{\text{avg}}}{ds} = \frac{m_{\text{flag}}}{2} \cdot \frac{d}{ds} [V^2]_{\text{avg}} \quad (4)$$

Integration of the remainder yields a relationship for average tension as a function of coordinate  $s$  along the rope

$$[T_{(s)}]_{\text{avg}} = \frac{m_{\text{flag}}}{2} \left\{ [V_{\text{tip}}^2]_{\text{avg}} - [V_{(s)}^2]_{\text{avg}} \right\} \quad (5)$$

and the average tension at the attachment point  $s = 0$ , where  $V = 0$ , is simply

$$[T_{s=0}]_{\text{avg}} = \frac{m_{\text{flag}}}{2} [V_{\text{tip}}^2]_{\text{avg}} \quad (6)$$

If we restrict the slope,  $(dw/dx)^2 \ll 1$ , we can insert the velocities from our wave motion

$$[V^2]_{\text{avg}} \cong \left[ \left( \frac{dw}{dt} \right)^2 \right]_{\text{avg}} = \frac{A^2 \omega^2}{2} \left( \frac{x}{L} \right)^2 \quad (7)$$

$$[T]_{\text{avg}} \cong \frac{m_{\text{flag}}}{2} \left[ \frac{A^2 \omega^2}{2} \right] \left\{ 1 - \left( \frac{x}{L} \right)^2 \right\} \quad (8)$$

$$[T_{x=0}]_{\text{avg}} \cong \frac{m_{\text{flag}}}{2} \left[ \frac{A^2 \omega^2}{2} \right] \quad (9)$$

We see that the average dynamically induced tension is distributed parabolically, decreasing from a maximum value at the flagpole to zero at the trailing edge. The gradient of the dynamically induced tension is negative, and its magnitude increases toward the trailing edge

$$\frac{d}{dx} [T]_{\text{avg}} \cong \frac{m_{\text{flag}}}{2} \left[ \frac{A^2 \omega^2}{2} \right] \left\{ \frac{-2x}{L^2} \right\} \quad (10)$$

With a great deal of computer-assisted algebra (Moretti, 2003) we can obtain the tension as a function of time; the bounds of the fluctuations are a modest fraction of the average tension if we get away from the trailing edge.

For a rough estimate of the average tension which acts to flatten the flag, we can use the *order-of-magnitude* of

$$\bar{T} \approx \frac{1}{6} \times m_{\text{flag}} A^2 \omega^2 \quad (11)$$

for the tension over the bulk of the flag area.

### 3 Flow Model

The structural model interacts with the flow model, which provides the destabilizing forces. By specifying a flow field which matches the structural boundary, we can compute the pressures at the boundary (Chang & Moretti, 2002). For potential flow, we can use a simplified approach in terms of effective mass  $m_{\text{air}}$  acting on the surface (Chang, Fox, Lilley, & Moretti, 1991). The pressure difference across the flag can be written in terms of the free-stream velocity  $U_\infty$  and the flag curvature .

$$\begin{aligned} \Delta p = - \left\{ \begin{aligned} & m_a \frac{\partial^2 w}{\partial t^2} + 2m_a U_\infty \frac{\partial^2 w}{\partial t \partial s} \left[ 1 + \left( \frac{\partial w}{\partial s} \right)^2 \right] + m_a U_\infty^2 \frac{\partial^2 w}{\partial s^2} \left[ 1 + \left( \frac{\partial w}{\partial s} \right)^3 \right] \\ & + 2U_\infty \int_s^L \frac{\partial w}{\partial s} \frac{\partial^2 w}{\partial t \partial s} ds + U_\infty^2 \int_s^L \frac{\partial w}{\partial s} \frac{\partial^2 w}{\partial s^2} ds \end{aligned} \right\} \\ \cong - \left\{ m_a \frac{\partial^2 w}{\partial t^2} + 2m_a U_\infty \frac{\partial^2 w}{\partial t \partial x} + m_a U_\infty^2 \frac{\partial^2 w}{\partial x^2} \right\} \quad (12) \end{aligned}$$

where  $m_a$  has a constant value of  $\pi\rho_{air}d/4$  for *slender* flags (width  $d$  much smaller than wavelength  $\lambda$ ), but is a function of wavelength  $\rho_{air}\lambda/\pi$  for *wide* flags (width  $d$  much greater than wavelength  $\lambda$ ).

If the air velocity is substantially greater than the phase velocity, we have *order-of-magnitude* balance of the destabilizing and the stabilizing forces when

$$\begin{aligned} m_a U_\infty^2 &= \frac{2}{3} \left( \frac{m}{2} \cdot \frac{A^2 \omega^2}{2} \right) \\ \text{wide flag } \frac{\rho_{air} \lambda}{\pi} U_\infty^2 &= \frac{1}{6} (mA^2 \omega^2) \\ \frac{A^2 \omega^2}{U_\infty^2} &= \frac{1}{6\pi} \frac{\rho_{air} \lambda}{m} \end{aligned} \quad (13)$$

This may be compared with experimental data, where  $A$ ,  $\omega$  and  $\lambda$  have been observed as a function of  $U_\infty$ . Data in the literature are uneven in terms of what information is reported

## 4 Experimental Data

Fairthorne (1930) conducted tests to determine the drag of banners towed by aircraft, at the end of a tow wire. He measured the drag of rectangular flags of length-to-width ratio 0.5–4.0, and of triangular flags, using different mass-per-unit-areas  $m$ . The speed range was 40–100 ft/sec. The shorter the tow (approx. 30 inch), the larger the drag. For fixed batten, fluctuations were approx. 30%. His conclusion was that the part of total drag not due to skin-friction varies directly with the mass-per-unit-area of the material, confirming the trend suggested by Equation 9. He noted that total drag decreases rapidly when increasing length-to-width ratio by slitting; this could be attributed to the change in the effective  $m_a$  between wide and slender flags mentioned above.

Fairthorne used the *dimensionless numbers* (with  $\approx$  order-of-magnitude):

- Fineness ratio of projected Area (= *length/width* for rectangular flags)

$$F \doteq Area \div (Width)^2 \approx 1.0$$

- Drag coefficient (referred to wetted area = both sides)

$$\begin{aligned} k_D \equiv c_D &\doteq \frac{Drag}{2 \times Area} \frac{2g_c}{U_\infty^2 \rho_{air}} \equiv \frac{\frac{1}{2} \bar{T}_o / Width}{L U_\infty^2 \rho_{air} / g_c} \\ &\approx \frac{12.4 \text{ lbf}}{\text{sec}} \left| \frac{\quad}{2 \times 16 \text{ ft}^2} \right| \left| \frac{2}{100^2 \text{ ft}^2} \right| \left| \frac{\text{ft}^3}{0.0765 \text{ lbm}} \frac{32.2 \text{ lbm-ft}}{\text{lbf-sec}^2} \right| \approx 0.033 \end{aligned}$$

- Density ratio

$$\sigma \doteq \frac{m_{flag}}{\rho_{air} \times Width} \approx \frac{0.016 \text{ lbm}}{\text{ft}^2} \left| \frac{\text{ft}^3}{0.0785 \text{ lbm}} \right| \frac{1}{4 \text{ ft}} \approx 0.0523$$

- Reynolds Number

$$\text{Re}_L \triangleq \frac{U_\infty L}{(\mu_{air} / \rho_{air})} \approx \frac{100 \text{ ft}}{\text{sec}} \left| \frac{4 \text{ ft}}{\quad} \right| \left| \frac{10^3 \text{ sec}}{0.16 \text{ ft}^2} \right| \approx 2.5 \times 10^6$$

- Viscous Drag

$$\text{laminar } c_D = 0.664 \text{Re}_L^{-1/2} \approx 0.664 \div \sqrt{2.5 \times 10^6} = 0.00042$$

$$\text{turbulent } c_D = 0.664 \text{Re}_L^{-1/5} \approx 0.03 \div \sqrt[5]{2.5 \times 10^6} = 0.0016$$

Göttingen data  $c_D = 0.012$  could be high at this  $\text{Re}_L$

Fairthorne's results featured a Table III. listing frequency of flutter for 4-ft-wide triangular flags as a function of Fineness ratio and wind speed:

<b>Frequency</b>	F=0.25	F=0.5	F=0.75	F=1.0
U=40 ft/sec	27.7 Hz	20.3 Hz	19.5 Hz	18.3 Hz
U=60 ft/sec	38.4 Hz	27.0 Hz	22.5 Hz	20.8 Hz
U=80 ft/sec		35.0 Hz	28.4 Hz	

Frequency increases moderately with wind speed, and increases also for very short flags. In the text, the amplitudes are reported greater for short flags.

Fairthorne's Table I. lists drag for 4-ft-wide rectangular flags at 100 ft/sec as a function of Fineness ratio:

<b>Drag</b>	F=0.48	F=1.0	F=1.5	F=2.0
$k_D$	0.057	0.033	0.025	0.020
$k_D - 0.012$	0.045	0.021	0.013	0.008
flutter-drag	8.2 lbf	7.9 lbf	7.4 lbf	6.0 lbf

We note that the total flutter-induced drag increases slightly for short flags, which is consistent with the idea that this drag is not proportional to the area, but to  $V_{end}^2 = \omega^2 A^2$  which are reported to increase moderately for short flags.

Fairthorne's Figure 1 shows the drag for square flags with two different basis weights,  $m_{flag} = 0.016$ , respectively  $0.032 \text{ lbf/ft}^2$ , as well as for a larger triangular silk flag of  $0.0075 \text{ lbf/ft}^2$

<b>Drag</b>	$m_{flag}=0.0075$	$m_{flag}=0.016$	$m_{flag}=0.032$
$k_D$	0.016	0.033	0.110
$k_D - 0.012$	0.004	0.021	0.098

Within the accuracy of the experiment and analysis, this is not inconsistent with the idea that the total flutter-induced drag is proportional to  $m_{flag}$ , but suggests that the drag may be proportional to an even higher power (e.g. the square) of the density ratio.

Fairthorne's Figure 2 compares the data with the semi-empirical correlation

$$\frac{k_D - 0.012}{\sigma} = 0.39F^{-1.25}$$

If we look at net drag, we multiply by the area and obtain

$$\frac{\overline{T}_o / \text{Width}}{\sigma} \propto F^{-0.25} \approx \text{constant}$$

leaving only a modest effect of the Fineness ratio, which could be explained by three-dimensional effects (flow around the sides of the flag).

Fairthorne's Table II. shows that the drag is somewhat reduced if the flag is slitted. This puzzling effect might also be explained by the change in  $m_a$  as suggested above: a reduction of frequency or amplitude would explain the reduction of the total drag.

We do not have enough information to predict the drag. From Thoma's analysis, the net drag of the mid-weight square flag could be explained by flutter in two-foot-long traveling waves with an estimated frequency of 30 Hertz (188 radians/second) — so that the wave-speed is 15 ft/sec — with a maximum r.m.s. amplitude of 0.6 ft.

$$[T_o]_{\text{avg}} = \frac{m_{\text{flag}}}{2} A^2 \omega^2 = \frac{1}{2} \left| \frac{0.016 \text{lbm}}{\text{ft}^2} \right| \left| \frac{\text{lbf} \cdot \text{sec}^2}{32.2 \text{lbm} \cdot \text{ft}} \right| \left| \frac{(0.6)^2 \text{ft}^2}{\text{sec}^2} \right| \frac{(188)^2}{\text{sec}^2} = \frac{12.4 \text{lbf}}{4 \text{ft}}$$

Hoerner (1958) cites Fairthorne on the increase of drag due to flutter, but attributes it to flow separation. Thoma's analysis shows that the assumption of flow separation is not required, nor is it sufficient to explain the conversion of normal forces into drag forces on a flexible flag.

Taneda (1968) experimentally observed no flutter at very low flow speeds; flapping motions as the flow speed is increased; regular oscillations in the range

$$10^4 < \text{Re}_L < 10^5$$

and violent and irregular oscillations at higher flow speeds. As shown in his Fig. 2, the frequency of regular flutter increases almost linearly with flow speed, is lower for heavy flags, and higher for short flags.

In stroboscopic pictures, Taneda shows not only fully fluttering flags, with wildly flapping trailing edges, in his Fig. 4A; but also flags with nodes in his Fig. 4D, where the motion upstream of the node is a traveling wave moving upstream. This cannot be explained by our fully flexible flag analysis, in which wave reflections from the trailing edge do not occur as they do in panels. Taneda also notes the three-dimensional effect of a corner rolling up in his Fig. 6, which deviates from our two-dimensional analysis.

Uno (1973) observed a growing traveling wave and compared it (unsuccessfully) with the vortex-shedding from a rigid panel. He believed that he had developed a rough estimate for the angular frequency and wave-number

$$\omega = 0.415 U_\infty \sqrt{\frac{\rho_{\text{air}}}{m_{\text{web}}} \cdot \frac{\text{width}}{L} \cdot (C_T + \text{const})}$$

$$\kappa = \frac{1.6911}{L} \text{ or } L^* = 1.6911 \text{ or } \lambda = \frac{2\pi L}{1.6911} \cong 3.7L$$

The latter indicates a total flag which is only slightly longer than a quarter-wave, as in his Fig. 2, while his stroboscopic figures show more complex shapes for all but the shortest flags.

Other experiments in the literature focused on critical speed, and bear only indirectly on post-critical behavior. Datta & Gottenberg (1975) investigated the critical speed of a flexible (mylar) panel tensioned by gravity and boundary-layer friction. In their experiments, panel stiffness proved to be unimportant. One of the ways they detected critical speed was by the sudden increase in measured drag force.

By analytical means, Yamaguchi, Yokota, & Tsujimoto (2000) reviewed numerical and variational approaches in the literature, and selected the assumption of a small perturbation on a two-dimensional finite-element tensioned plate, subjected to potential flow, in order to predict critical speed, as well as frequencies and modes of the initial oscillations.

In related experiments, Yamaguchi, Sekiguchi, Yokota, & Tsujimoto (2000) measured critical velocities. To display results, they used:

- The ratio of mass-per-unit-area  $m_{flag}$  compared to the air density  $\rho_{air}$  and flag length  $L$

$$\text{mass ratio } \mu \triangleq \frac{m_s}{\rho_{air}L}$$

with low values, ranging from 0.02 to 0.6, supplemented by wider-ranging data (0.01–1.0) from Watanabe (1995) and higher- $\mu$  data (1.0–3.0) from Huang (1995).

- The “relative stiffness” of the panel

$$\beta = \frac{EI}{\frac{1}{2}\rho_{air}U_{crit}^2L^3}$$

an unfortunate choice for flags, where the stiffness-effect is small compared to tension-effects.

- A Strouhal number

$$f_{\text{Red.}} = \frac{fL}{U_{crit}}$$

- and the friction factor  $c_f$ , arbitrarily assumed to have values in the range of 0.05 to 0.15.

Contrary to the authors’ assertion, the fit of experiment to prediction is poor; furthermore, there are marked differences between the results from different sheet materials.

## 5 Conclusions

The available flexible-flag experiments are generally consistent with the hypothesis that dynamically induced tension is the dominant term for prediction the post-critical amplitude of flag flutter, but the information given is insufficient to prove it. We are planning a series of wind-tunnel experiments in which not only critical speed, but also amplitudes, frequencies, wave-lengths, and standing-wave-ratio will be measured, for a more definitive answer.

Where there are significant differences between observations and the patterns predicted by our hypothesis, other factors such as panel stiffness appear to be present. Conversely, the inclusion of in-plane kinetic energy in cantilevered-panel flutter analysis might improve the prediction of post-critical amplitudes of flexible panels.

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