Complex-number Modes II.
Damped multi-degree-of-freedom Systems

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1 Problem Statement

Is it possible to express the modes of arbitrarily (non-Rayleigh) damped multi-degree-of-freedom systems like classical modes, but with phase shifts — analogous to expressing a solution of the thread-line equation as

$$y_1 = A_1 \operatorname{Re}\left\{\sin\xi\left[\cos\beta\xi + i\sin\beta\xi\right]e^{i\left(1-\beta^2\right)t}\right\}$$

which is a harmonic function of time $\sin\omega t$, multiplied by a spatial mode consisting of wave with a phase-shift

$$Y_1 = \sin\xi\left[\cos\beta\xi + i\sin\left(\xi\right)\right]$$

This would have didactic advantages in treating damped modes as complex-number extensions of classical modes.

Example: Investigate two identical boxcars m with springs k to ground, a coupling-spring $1.5k$, and one damper $c = 1.0\sqrt{km}$ from the first mass to ground. Scaling time to $\sqrt{k/m}$, the normalized governing equation is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} x_1'' \\ x_2'' \end{Bmatrix} + \begin{bmatrix} 1.0 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} x_1' \\ x_2' \end{Bmatrix} + \begin{bmatrix} 2.5 & -1.5 \\ -1.5 & 2.5 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1}$$

or, in state-variable form

$$\begin{Bmatrix} x_1' \\ x_2' \\ v_1' \\ v_2' \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2.5 & +1.5 & -1.0 & 0 \\ +1.5 & -2.5 & 0 & 0 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{Bmatrix} \tag{2}$$

The characteristic equation is

$$\begin{aligned} s^4 + 1.0s^3 + 5.0s^2 + 2.5s + 4.0 &= 0 \\ \left(s^2 + 0.548s + 1.093\right)\left(s^2 + 0.452s + 3.659\right) &= 0 \\ (s + .274 - 1.009i)(s + .274 + 1.009i)(s + .226 - 1.899i)(s + .226 + 1.899i) &= 0 \end{aligned}$$

$$i\omega = s_{a,b,c,d} = -.274 \pm 1.009i \text{ and } -.226 \pm 1.900i, \text{ respectively} \tag{3}$$

We therefore know that the governing equation for the principal coordinates is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} p_I'' \\ p_{II}'' \end{array} \right\} + \begin{bmatrix} 0.548 & 0 \\ 0 & 0.452 \end{bmatrix} \left\{ \begin{array}{c} p_I' \\ p_{II}' \end{array} \right\} + \begin{bmatrix} 1.093 & 0 \\ 0 & 3.659 \end{bmatrix} \left\{ \begin{array}{c} p_I \\ p_{II} \end{array} \right\} \tag{4}$$

Question: How do we transform from the x-coordinates in Equation 1 to these principal-coordinates p_I and p_{II} ?

In state-variable form, the procedure is easy but laborious: the transformed equation is

$$\left\{ \begin{array}{c} p_a' \\ p_b' \\ p_c' \\ p_d' \end{array} \right\} = \begin{bmatrix} -.274+1.009i & 0 & 0 & 0 \\ 0 & -.274-1.009i & 0 & 0 \\ 0 & 0 & -.226+1.899i & 0 \\ 0 & 0 & 0 & -.226-1.899i \end{bmatrix} \left\{ \begin{array}{c} p_a \\ p_b \\ p_c \\ p_d \end{array} \right\}$$

where the initial values of the principal coordinates $p_{a,b,c,d}$ are obtained from $\{p_{(0)}\} = [U^{-1}]\{x_{(0)}\}$, and the results are transformed back to the original coordinates $\{x_{(t)}\} = [U]\{p_{(t)}\}$; the transformation matrices are obtained from the eigenvectors of Equation 2

$$[U] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.855+0.304i & 0.855-0.304i & -0.855+0.694i & -0.855-0.694i \\ -0.274+1.009i & -0.274-1.009i & -0.226+1.900i & -0.226-1.900i \\ -0.541+0.780i & -0.541-0.780i & -1.126-1.782i & -1.126+1.782i \end{bmatrix}$$

$$[U^{-1}] = \begin{bmatrix} 0.235-0.314i & 0.286+0.089i & -0.094-0.301i & 0.011-0.286i \\ 0.235+0.314i & 0.286-0.089i & -0.094+0.301i & 0.011+0.286i \\ 0.265+0.101i & -0.286-0.055i & 0.094-0.101i & -0.011+0.152i \\ 0.265-0.101i & -0.286+0.055i & 0.094+0.101i & -0.011-0.152i \end{bmatrix}$$

We can check that these eigenvectors diagonalize the characteristic matrix into the spectral matrix

$$[U^{-1}] \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2.5 & +1.5 & -1.0 & 0 \\ +1.5 & -2.5 & 0 & 0 \end{bmatrix} [U]$$

$$\Rightarrow \begin{bmatrix} -.274+1.009i & 0 & 0 & 0 \\ 0 & -.274-1.009i & 0 & 0 \\ 0 & 0 & -.226+1.9i & 0 \\ 0 & 0 & 0 & -.226-1.9i \end{bmatrix}$$

and we obtain the complex-number solutions for p

$$\left\{ \begin{array}{c} p_a \\ p_b \\ p_c \\ p_d \end{array} \right\} = \left\{ \begin{array}{c} p_{a(0)} \times e^{(-.274+1.009i)t} \\ p_{b(0)} \times e^{(-.274-1.009i)t} \\ p_{c(0)} \times e^{(-.226+1.899i)t} \\ p_{d(0)} \times e^{(-.226-1.899i)t} \end{array} \right\}$$

which will ultimately transform into real-number solutions $\{x\} = [U]\{p\}$. However, this round-about procedure is didactically unattractive, because the meaning of the complex-number principal coordinates is difficult to explain.

We would like to transform directly between x-coordinates and real-number principal coordinates solutions

$$\left\{ \begin{array}{c} p_I \\ p_{II} \end{array} \right\} = \left\{ \begin{array}{c} e^{-.274t^*} \left(A \sin 1.009t^* + B \cos 1.009t^*\right) \\ e^{-.226t^*} \left(C \sin 1.900t^* + D \cos 1.900t^*\right) \end{array} \right\}$$

The mode-shapes x_2/x_1 for each mode are the same as in $[U]$

$$[u] = \left[\left\{ \begin{array}{c} 1 \\ 0.855 \pm 0.304i \end{array} \right\}_I \left\{ \begin{array}{c} 1 \\ -0.855 \pm 0.694i \end{array} \right\}_{II} \right]$$

where the complex numbers indicate phase-angles of

$$\begin{aligned} \theta_I &= \pm \arctan\left(0.304/0.855\right) = \pm 0.341 \text{ radians} \\ \theta_{II} &= \pm \arctan\left(0.694/0.855\right) = \pm 0.682 \text{ radians} \end{aligned}$$

and the relative amplitudes are

$$\begin{aligned} \frac{|x_2|}{|x_1|} &= \sqrt{0.855^2 + 0.304^2} = 0.907 \\ \text{respectively} &= \sqrt{0.855^2 + 0.694^2} = 1.101 \end{aligned}$$

allowing us to express the relative motions in terms of phasors. The problem is that we are given a choice of signs: *Which of the alternative signs are meaningful?*

It appears that, if we insert the solutions

$$\begin{aligned} \frac{x_2}{x_1} &= \frac{0.855 \cos\left(\omega_I t^*\right) \pm 0.304 \sin\left(\omega_I t^*\right)}{1.0 \cos\left(\omega_I t^*\right)} = 0.855 \pm 0.304 \tan\left(\omega_I t^*\right) \\ \text{or} \quad &= \frac{0.855 \cos\left(\omega_{II} t^*\right) \pm 0.694 \sin\left(\omega_{II} t^*\right)}{1.0 \cos\left(\omega_{II} t^*\right)} = 0.855 \pm 0.694 \tan\left(\omega_{II} t^*\right) \end{aligned}$$

into the original Equation 1, we discover that the minus signs (related to $s_b = (-.274 - 1.009i)$ and $s_d = (-.226 - 1.899i)$) satisfy the equation, while the plus signs (related to $s_a = (-.274 + 1.009i)$ and $s_c = (-.226 + 1.899i)$) do not satisfy the equation — *Why? Could I have predicted this?*

If we accept the description of the modes

$$\begin{aligned} [u] &= \begin{bmatrix} 1 & 1 \\ 0.855 - 0.304i & -0.855 - 0.694i \end{bmatrix} \\ \left[u^{-1}\right] &= \begin{bmatrix} 0.563 + 0.277i & 0.556 - 0.127i \\ 0.437 - 0.277i & -0.556 + 0.127i \end{bmatrix} \end{aligned}$$

How do we apply this, to transform from

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} x_1'' \\ x_2'' \end{array} \right\} + \begin{bmatrix} 1.0 & 0 \\ 0 & 0 \end{bmatrix} \left\{ \begin{array}{c} x_1' \\ x_2' \end{array} \right\} + \begin{bmatrix} 2.5 & -1.5 \\ -1.5 & 2.5 \end{bmatrix} \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\}$$

to the diagonalized

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} p_I'' \\ p_{II}'' \end{array} \right\} + \begin{bmatrix} 0.548 & 0 \\ 0 & 0.452 \end{bmatrix} \left\{ \begin{array}{c} p_I' \\ p_{II}' \end{array} \right\} + \begin{bmatrix} 1.093 & 0 \\ 0 & 3.659 \end{bmatrix} \left\{ \begin{array}{c} p_I \\ p_{II} \end{array} \right\}$$

The complex-number modal matrix in this form does **not** directly diagonalize the damping and stiffness matrices!

2 Systematic Solution

Writing solutions in a form similar to that which worked for the thread-line equation

$$
\begin{aligned}
\left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\}_I &= A_I e^{-.274t^*} \left\{ \begin{array}{c} 1.0 \cos(1.009t^*) \\ 0.855 \cos(1.009t^*) - 0.304 \sin(1.009t^*) \end{array} \right\} \\
&= A_I e^{-.274t^*} \left\{ \begin{array}{c} \mathrm{Re}\left(1.0 e^{i1.009t^*}\right) \\ \mathrm{Re}\left(0.855 e^{i1.009t^*} + i0.304 e^{i1.009t^*}\right) \end{array} \right\} \\
&= A_I e^{-.274t^*} \left\{ \begin{array}{c} \mathrm{Re}\left(1.0 e^{i1.009t^*}\right) \\ \mathrm{Re}\left(0.907 e^{i(1.009t^* - 0.341)}\right) \end{array} \right\} \\
&= A_I e^{-.274t^*} \left\{ \begin{array}{c} 1.0 \cos(1.009t^*) \\ 0.907 \cos(1.009t^* - 0.341) \end{array} \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\}_{II} &= A_{II} e^{-.226t^*} \left\{ \begin{array}{c} 1.0 \cos(1.9t^*) \\ -0.855 \cos(1.9t^*) - 0.694 \sin(1.9t^*) \end{array} \right\} \\
&= A_{II} e^{-.226t^*} \left\{ \begin{array}{c} \mathrm{Re}\left(1.0 e^{i1.9t^*}\right) \\ \mathrm{Re}\left(0.855 e^{i1.9t^*} + i0.694 e^{i1.9t^*}\right) \end{array} \right\} \\
&= A_{II} e^{-.226t^*} \left\{ \begin{array}{c} \mathrm{Re}\left(1.0 e^{i1.9t^*}\right) \\ \mathrm{Re}\left(1.101 e^{i(1.9t^* - 0.682)}\right) \end{array} \right\} \\
&= A_{II} e^{-.226t^*} \left\{ \begin{array}{c} 1.0 \cos(1.9t^*) \\ 1.101 \cos(1.9t^* - 0.682) \end{array} \right\}
\end{aligned}
$$

we can write the modal matrix dynamically

$$
\begin{aligned}
[u] &\propto \left[\begin{array}{cc} 1.0 \cos(1.009t) & 1.0 \cos(1.9t) \\ 0.855 \cos(1.009t) - 0.304 \sin(1.009t) & -0.855 \cos(1.9t) - 0.694 \sin(1.9t) \end{array} \right] \\
\left[\frac{du}{dt}\right] &\propto \left[\begin{array}{cc} -0.274 \cos(1.009t) - 1.009 \sin(1.009t) & -0.226 \cos(1.9t) - 1.900 \sin(1.9t) \\ -0.541 \cos(1.009t) - 0.779 \sin(1.009t) & -1.125 \cos(1.9t) + 1.781 \sin(1.9t) \end{array} \right] \\
\left[\frac{d^2u}{dt^2}\right] &\propto \left[\begin{array}{cc} -0.943 \cos(1.009t) + 0.553 \sin(1.009t) & -3.559 \cos(1.9t) + 0.859 \sin(1.9t) \\ -0.638 \cos(1.009t) + 0.759 \sin(1.009t) & 3.639 \cos(1.9t) + 1.736 \sin(1.9t) \end{array} \right]
\end{aligned}
$$

and insert it into Equation 1

$$
\left\{ \begin{array}{c} \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \left[\frac{d^2u}{dt^2}\right] \\ + \left[\begin{array}{cc} 1.0 & 0 \\ 0 & 0 \end{array} \right] \left[\frac{du}{dt}\right] \\ + \left[\begin{array}{cc} 2.5 & -1.5 \\ -1.5 & 2.5 \end{array} \right] [u] \end{array} \right\} = \left[\begin{array}{cc} 0.0 & 0.0 \\ 0.0 & 0.0 \end{array} \right]
$$

The problem remains: how do we obtain the diagonalization of Equation 1 without the detour through solution through either the Laplace transform or else the state-variable formulation?

3 Addendum: Mathematical Theory

The mathematical basis is as follows:[1] Consider

$$[M]\,\ddot{x} + [D]\,\dot{x} + [C]\,x = 0$$

where $M \in R^{n\times n}$ etc., and the eigenvalue problem obtained with

$$x = ue^{\lambda t}$$

The eigenpairs $u_p.\lambda_p.$ for $p = 1\ldots 2n$ occur in complex conjugate pairs. Let

$$u = \left\{\begin{array}{c} u_1 e^{i\beta_1} \\ u_2 e^{i\beta_2} \\ \vdots \\ \vdots \\ u_n e^{i\beta_n} \end{array}\right\} .\lambda = -\delta + i\omega \text{ and } u = \left\{\begin{array}{c} u_1 e^{-i\beta_1} \\ u_2 e^{-i\beta_2} \\ \vdots \\ \vdots \\ u_n e^{-i\beta_n} \end{array}\right\} .\lambda = -\delta - i\omega$$

be one such pair. Consider the particular solution to the original equation

$$x = K\left(e^{\lambda t}u\right) + K^*\left(e^{\lambda t}u\right)^*$$

where $K = ke^{i\gamma}$ is an arbitrary constant of integration, and * stands for "complex conjugate." This gives the real solution

$$x = 2ke^{-\delta t}\left\{\begin{array}{c} u_1\cos\left(\omega t + \gamma + \beta_1\right) \\ u_2\cos\left(\omega t + \gamma + \beta_2\right) \\ \vdots \\ \vdots \\ u_n\cos\left(\omega t + \gamma + \beta_n\right) \end{array}\right\}$$

with real constants of integration k and γ, all the other parameters being "modal." A relation between a "principal coordinate solution" and the solution x given above has to involve a different phase-shift β for each coordinate; *i.e.,* in the real representation the transformation should contain a different rotation for each component.

[1] contributed July 2003 by Prof. Peter Hagedorn, FB6 Mechanik/Dynamik AG2, Technische Universität Darmstadt, Hochschulstraße 1, D-64289 Darmstadt, Germany.