Complex-number Modes I.
Gyroscopic Equations

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1 Governing Equation

Consider a second-order P.D.E. with a mixed-derivative term

$$a\frac{\partial^2 y}{\partial t^2} \pm 2b\frac{\partial^2 y}{\partial t \partial x} - c\frac{\partial^2 y}{\partial x^2} = 0 \tag{1}$$

with the End Conditions $y = 0$ at $x = 0$ and at $x = \ell$. Such equations govern translating threadlines and submerged webs[1]. If c is positive (a requirement for a non-divergent solution), we transform this into the Canonical Form

$$\frac{\partial^2 \eta}{\partial \tau^2} + 2\beta\frac{\partial^2 \eta}{\partial \tau \partial \xi} - \left(1 - \beta^2\right)\frac{\partial^2 \eta}{\partial \xi^2} = 0 \tag{2}$$

in terms of dimensionless parameters

$$\begin{aligned}
\eta &\doteq \frac{y}{L} \\
\tau &\doteq \frac{\pi\sqrt{b^2 + ac}}{a\ell}t \\
\beta &\doteq \sqrt{\frac{b^2}{b^2 + ac}} < 1 \\
\xi &\doteq \pm\frac{\pi x}{\ell}
\end{aligned}$$

where the choice of coordinate direction ξ allows us to make the middle term positive, and the End Conditions are $\eta = 0$ at $\xi = 0$ and at $\xi = \pi$.

We can compare this equation with the Wave Equation (with the time scale normalized to make $c^2 = 1$), and see that this is the the same equation with a

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[1] Y.B. Chang, S.J. Fox, D.G. Lilley, & P.M. Moretti, “Aerodynamics of Moving Belts, Tapes, and Webs,” DE-Vol. 36, *Machinery Dynamics and Element Vibrations,* pages 33–40, ASME Design Conference, Miami, Florida, Sept. 22–25, 1991, ISBN No. 0-7918-0627-8, pages 33–40; full-text on http://www.mae.okstate.edu/Faculty/moretti/moretti.html

translating coordinate system. Since we know that the Wave Equation is satisfied by wave solutions with phase velocities c and $-c$, we can predict from coordinate transformation,[2] that this governing equation is satisfied by *traveling-wave* solutions

$$\begin{aligned} \eta_{(\xi,\tau)} &= \mathcal{F}_1(\xi - \tau - \beta\tau) + \mathcal{F}_2(\xi + \tau - \beta\tau) \\ &= \mathcal{F}_1\left(\xi - (1+\beta)\,\tau\right) + \mathcal{F}_2\left(\xi + (1-\beta)\,\tau\right) \end{aligned} \tag{3}$$

so that the downstream phase-velocity is $(1+\beta)$ and the upstream phase-velocity is $(1-\beta)$.

2 First Solution

To meet the End Conditions, we have to superpose downstream- and upstream-traveling waves of the same frequency $n\left(1-\beta^2\right)$, but different wavelengths $2\pi/n\,(1-\beta)$ and $2\pi/n\,(1+\beta)$

$$\begin{aligned} \eta_{(\xi,\tau)} &= \left\{ \begin{array}{l} \frac{A_n}{2}\left[\sin\left(n\,(1-\beta)\,(\xi-(1+\beta)\,\tau)\right) + \sin\left(n\,(1+\beta)\,(\xi+(1-\beta)\,\tau)\right)\right] \\ +\frac{B_n}{2}\left[\cos\left(n\,(1-\beta)\,(\xi-(1+\beta)\,\tau)\right) - \cos\left(n\,(1+\beta)\,(\xi+(1-\beta)\,\tau)\right)\right] \end{array} \right\} \\ \eta_{(0,\tau)} &= 0 = \eta_{(\pi,\tau)} \end{aligned} \tag{4}$$

beating against each other to create a new wave-length $2\pi/n\beta$, and an *oscillation envelope* or modulation in space $\sin(n\xi)$.

3 Second Solution

Because of the mixed derivative in this P.D.E., separation of the variables ξ and τ is not possible; adding the traveling waves gives us the well-known solution[3]

$$\begin{aligned} \eta_{(\xi,\tau)} &= \left\{ \begin{array}{l} \sum_{n=1}^{\infty} A_n \sin(n\xi)\cos\left(n\beta\xi + n\left(1-\beta^2\right)\tau\right) \\ +\sum_{n=1}^{\infty} B_n \sin(n\xi)\sin\left(n\beta\xi + n\left(1-\beta^2\right)\tau\right) \end{array} \right\} \\ \frac{\partial\eta}{\partial\tau} &= \left\{ \begin{array}{l} \sum_{n=1}^{\infty} -A_n n\left(1-\beta^2\right)\sin(n\xi)\sin\left(n\beta\xi + n\left(1-\beta^2\right)\tau\right) \\ +\sum_{n=1}^{\infty} +B_n n\left(1-\beta^2\right)\sin(n\xi)\cos\left(n\beta\xi + n\left(1-\beta^2\right)\tau\right) \end{array} \right\} \end{aligned} \tag{5}$$

The coefficient $\sin(n\xi)$ assures us that the End Conditions are met. Within this envelope, the motion is a traveling wave moving *upstream* with phase-velocity $\left(1-\beta^2\right)/\beta$. The amplitude coefficients A_n and B_n come from the Initial Conditions

$$\begin{aligned} \eta_{(\xi,0)} &= \left\{ \begin{array}{l} \sum_{n=1}^{\infty} A_n \sin(n\xi)\cos(n\beta\xi) \\ +\sum_{n=1}^{\infty} B_n \sin(n\xi)\sin(n\beta\xi) \end{array} \right\} \\ \left[\frac{\partial\eta}{\partial\tau}\right]_{\tau=0} &= \left\{ \begin{array}{l} \sum_{n=1}^{\infty} -A_n n\left(1-\beta^2\right)\sin(n\xi)\sin(n\beta\xi) \\ +\sum_{n=1}^{\infty} +B_n n\left(1-\beta^2\right)\sin(n\xi)\cos(n\beta\xi) \end{array} \right\} \end{aligned}$$

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[2] *see* http://www.mae.okstate.edu/Faculty/moretti/TransWav.pdf

[3] Peter M. Moretti, *Modern Vibrations Primer,* CRC Press, Boca Raton, FL 33431, ©2000, ISBN 0-8493-2038-0, page 302.

4 Phasor notation

Recalling the relationships

$$\begin{aligned}
e^{\pm i\omega t} &= \cos\omega t \pm i\sin\omega t \\
\cos(\omega t) &= \operatorname{Re}\left(e^{\pm i\omega t}\right) \\
\pm\sin(\omega t) &= \operatorname{Im}\left(e^{\pm i\omega t}\right)
\end{aligned}$$

we express phase in a traveling wave; *e.g.,* in the upstream direction, by writing

$$\begin{aligned}
y &= \cos(\kappa x + \omega t) \\
&= \cos\kappa x\cos\omega t - \sin\kappa x\sin\omega t \\
&= \cos(\kappa x)\operatorname{Re}\left(e^{i\omega t}\right) - \sin(\kappa x)\operatorname{Im}\left(e^{i\omega t}\right) \\
&= \cos(\kappa x)\operatorname{Re}\left(e^{i\omega t}\right) + \sin(\kappa x)\operatorname{Re}\left(ie^{i\omega t}\right) \\
&= \operatorname{Re}\left(\left[\cos\kappa x + i\sin\kappa x\right]e^{i\omega t}\right)
\end{aligned}$$

The variables x and t are separated: $[\cos\kappa x + i\sin\kappa x]$ represents the mode shape, with the real-part representing in-phase motion, and the imaginary part the component of the motion which is ninety degrees out-of-phase in time.

If we sketch out the motion of the fundamental mode of Equation 5

$$\eta = A_1\sin(\xi)\cos\left(\beta\xi + \left(1-\beta^2\right)\tau\right)$$

we find that the web has a harmonic motion just like a classic mode, but that there is a phase-difference between the motions at different stations along the web span. To express this in the form of a classic mode, we again use complex numbers for the amplitude distribution

$$\eta = A_1 \times \operatorname{Re}\left(\left[\sin(\xi)\cos(\beta\xi) + i\sin(\xi)\sin(\beta\xi)\right]e^{i\left(1-\beta^2\right)\tau}\right)$$

where the bracket describes the mode shape using complex numbers. the mode shape.

5 Third Solution

Therefore we can write the solution

$$\eta_{(\xi,\tau)} = \left\{\begin{array}{c} \sum_{n=1}^{\infty} A_n\sin(n\xi)\operatorname{Re}\left(\left[\cos(n\beta\xi) + i\sin(n\beta\xi)\right]e^{in\left(1-\beta^2\right)\tau}\right) \\ +\sum_{n=1}^{\infty} B_n\sin(n\xi)\operatorname{Re}\left(\left[\cos(n\beta\xi) + i\sin(n\beta\xi)\right]ie^{in\left(1-\beta^2\right)\tau}\right) \end{array}\right\} \tag{6}$$

$$\frac{\partial\eta}{\partial\tau} = \left\{\begin{array}{c} \sum_{n=1}^{\infty} -A_n n\left(1-\beta^2\right)\sin(n\xi)\operatorname{Re}\left(\left[\cos(n\beta\xi) + i\sin(n\beta\xi)\right]ie^{in\left(1-\beta^2\right)\tau}\right) \\ +\sum_{n=1}^{\infty} B_n n\left(1-\beta^2\right)\sin(n\xi)\operatorname{Re}\left(\left[\cos(n\beta\xi) + i\sin(n\beta\xi)\right]e^{in\left(1-\beta^2\right)\tau}\right) \end{array}\right\}$$

We have separated the variables ξ and τ by using the complex-number mode-shapes

$$Y_n = \sin(n\xi)\left[\cos(n\beta\xi) + i\sin(n\beta\xi)\right] \tag{7}$$

This was possible because we can represent the functions of time in the Second Solution as $\sin\left(n\left(1-\beta^2\right)\tau+\theta\right)$ or $\cos\left(n\left(1-\beta^2\right)\tau+\theta\right)$, where the phase angle $\theta = n\beta\xi$ is only a function of position ξ.

Addendum: Method of Solution

The solutions given here could have been obtained by classical methods.[4] For this homogeneous linear PDE with constant coefficients, consider a harmonic solution of the form $\eta = W_{(\xi)}e^{i\lambda t}$. Substituting in Equation 2 yields

$$\begin{aligned}\left[-\lambda^2 W + i2\lambda\beta W' - \left(1-\beta^2\right)W''\right]e^{i\lambda t} &= 0\\ \left(\beta^2-1\right)W'' + i2\lambda\beta W' - \lambda^2 W &= 0\end{aligned}$$

The solution of this ODE can be taken in the form $W_{(\xi)} = \alpha e^{ikx}$, yielding the characteristic equation

$$\begin{aligned}\alpha\left[-\left(\beta^2-1\right)k^2 - 2\lambda\beta k - \lambda^2\right]e^{ikx} &= 0\\ \left(1-\beta^2\right)k^2 - 2\lambda\beta k - \lambda^2 &= 0\end{aligned}$$

with the roots $k_1 = -\lambda/\left(1+\beta\right)$ and $k_2 = \lambda/\left(1-\beta\right)$; thus,

$$W = \frac{A}{2}e^{-i\frac{\lambda}{1+\beta}\xi} + \frac{B}{2}e^{i\frac{\lambda}{1-\beta}\xi}$$

It is to be noted that, in a complex solution of a real linear differential equation, both the real part and the imaginary part satisfy the differential equation. One can write the solution as

$$\eta = \left[\frac{A}{2}e^{-i\frac{\lambda}{1+\beta}\xi} + \frac{B}{2}e^{i\frac{\lambda}{1-\beta}\xi}\right]e^{ikx} = \eta_{\mathrm{Re}} + i\eta_{\mathrm{Im}}$$

where

$$\begin{aligned}\eta_{\mathrm{Re}} &= \left\{\begin{array}{l}\left[\frac{A}{2}\cos\left(\frac{\lambda}{1+\beta}\xi\right) + \frac{B}{2}\cos\left(\frac{\lambda}{1-\beta}\xi\right)\right]\cos\lambda t\\ -\left[\frac{A}{2}\sin\left(\frac{\lambda}{1+\beta}\xi\right) + \frac{B}{2}\sin\left(\frac{\lambda}{1-\beta}\xi\right)\right]\sin\lambda t\end{array}\right\}\\ \eta_{\mathrm{Im}} &= \left\{\begin{array}{l}\left[\frac{A}{2}\cos\left(\frac{\lambda}{1+\beta}\xi\right) + \frac{B}{2}\cos\left(\frac{\lambda}{1-\beta}\xi\right)\right]\sin\lambda t\\ +\left[\frac{A}{2}\sin\left(\frac{\lambda}{1+\beta}\xi\right) + \frac{B}{2}\sin\left(\frac{\lambda}{1-\beta}\xi\right)\right]\cos\lambda t\end{array}\right\}\end{aligned}$$

The End Conditions require

$$\begin{aligned}A + B &= 0\\ Ae^{-i\frac{\lambda}{1+\beta}\pi} + Be^{i\frac{\lambda}{1-\beta}\pi} &= 0\end{aligned}$$

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[4] contributed July 2003 by Prof. Anirvan Dasgupta, Dept. of Mechanical Engineering, IIT Kharagpur – 721 302, India; with suggestions from Prof. Peter Hagedorn, FB6 Mechanik/Dynamik AG2, Technische Universität Darmstadt, Hochschulstraße 1, D-64289 Darmstadt, Germany.

Using the left-end condition $A + B = 0$ to eliminate B yields on simplification

$$\begin{aligned}
\eta_{\mathrm{Re}} &= A\left[\cos\left(\frac{\lambda\xi}{1+\beta}\right) - \cos\left(\frac{\lambda\xi}{1-\beta}\right)\right]\cos\lambda\tau + A\left[\sin\left(\frac{\lambda\xi}{1+\beta}\right) + \sin\left(\frac{\lambda\xi}{1-\beta}\right)\right]\sin\lambda\tau \\
&= A\cos\left(\frac{\lambda\xi}{1+\beta} - \lambda\tau\right) - A\cos\left(\frac{\lambda\xi}{1+\beta} + \lambda\tau\right) \\
&= A\cos\left(k_1\left[\xi - \left(1+\beta\right)\tau\right]\right) - A\cos\left(k_2\left[\xi + \left(1-\beta\right)\tau\right]\right) \\
&= \mathcal{F}_1\left(\xi - \left(1+\beta\right)\tau\right) + \mathcal{F}_2\left(\xi + \left(1-\beta\right)\tau\right)
\end{aligned}$$

Using the right-end condition yields

$$\begin{aligned}
e^{-i\frac{\lambda}{1+\beta}\pi} - e^{i\frac{\lambda}{1-\beta}\pi} &= 0 \\
e^{-i\frac{\lambda}{1+\beta}\pi}\left(1 - e^{i\frac{2\lambda}{1-\beta^2}\pi}\right) &= 0 \\
e^{i\frac{2\lambda}{1-\beta^2}\pi} &= 1 \\
\lambda_n &= n\left(1-\beta^2\right)
\end{aligned}$$

Combining the end-condition solutions one finds

$$\begin{aligned}
\eta &= A_n\cos\left(n\left(1-\beta\right)\xi - \lambda_n\tau\right) - A_n\cos\left(n\left(1+\beta\right)\xi + \lambda_n\tau\right) \\
&= 2A_n\sin\left(n\xi\right)\sin\left(n\beta\xi + \lambda_n\tau\right)
\end{aligned}$$

It may be observed that the solution is composed of a backwards-traveling wave whose amplitude is modulated by a sine-function envelope.