# *Complex-number Modes III.*
# Damped Strings
# *Working Paper*

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## 1 Velocity damping

A string with linear velocity-damping (and possibly slight stiffness) has the normalized governing equation

$$\frac{\partial^2 y}{\partial t^2} + 2b\frac{\partial y}{\partial t} - c^2\frac{\partial^2 y}{\partial x^2} + \varepsilon^4\frac{\partial^4 y}{\partial x^4} = 0 \tag{1}$$

with pinned or clamped end conditions at $x = 0$ and at $x = \ell$. With the heuristic *Ansatz*

$$y_{(t,x)} = G_{(t)} \times Y\,(x) \tag{2}$$

we can separate variables and obtain two ordinary differential equations

$$\frac{d^2G}{dt^2} + 2b\frac{dG}{dt} + \omega^2 G \quad = \quad 0 \tag{3}$$

$$\frac{\varepsilon^4}{c^2}\frac{d^4Y}{dx^4} - \frac{d^2Y}{dx^2} - \frac{\omega^2}{c^2}Y \quad = \quad 0 \tag{4}$$

the first of which tells us that we have exponentially decaying oscillations, and the solutions to to the second are **classical modes** $Y_n$ of standing waves in the form of trigonometric and/or hyperbolic functions. For example, if the end conditions are simple supports, the modes are

$$Y_n = \sin\frac{n\pi x}{\ell} \tag{5}$$

which we plug into the Equation 4 to find

$$\left[\frac{\varepsilon^4}{c^2}\left(\frac{n\pi}{\ell}\right)^4 + \left(\frac{n\pi}{\ell}\right)^2 - \frac{\omega_n^2}{c^2}\right]\sin\frac{n\pi x}{\ell} = 0 \tag{6}$$

giving us the resonance frequencies

$$2\pi f_n = \omega_n = \sqrt{\left(\frac{n\pi c}{\ell}\right)^2 + \left(\frac{n\pi\varepsilon}{\ell}\right)^4} \tag{7}$$

Plugging these nominal frequencies into the Equation 3 we obtain the decay rate and the actual damped oscillation frequencies

$$G_n = e^{-bt}\left[A_n \sin\left(t\sqrt{\omega_n^2 - b^2}\right) + B_n \cos\left(t\sqrt{\omega_n^2 - b^2}\right)\right] \tag{8}$$

This standing wave can also be represented by the superposition of traveling waves

$$y_n = e^{-bt}\left\{\begin{array}{l} \frac{A_n}{2}\left[\sin\left(t\sqrt{\omega_n^2 - b^2} - \frac{n\pi}{\ell}x\right) - \sin\left(t\sqrt{\omega_n^2 - b^2} + \frac{n\pi}{\ell}x\right)\right] \\ \frac{B_n}{2}\left[\cos\left(t\sqrt{\omega_n^2 - b^2} - \frac{n\pi}{\ell}x\right) - \cos\left(t\sqrt{\omega_n^2 - b^2} + \frac{n\pi}{\ell}x\right)\right] \end{array}\right\} \tag{9}$$

These waves have phase velocities

$$\begin{aligned} v_n &= \left(\frac{2\ell}{n}\right)\frac{1}{2\pi}\sqrt{\omega_n^2 - b^2} \\ &= \sqrt{c^2 + \left(\frac{n\pi}{\ell}\right)^2 \varepsilon^4 - b^2} \end{aligned} \tag{10}$$

We see that waves are dispersive if there is any stiffness in the string.

# 2 Bending damping

A string with linear damping arising from bending the material of the string leads to the normalized governing equation

$$\frac{\partial^2 y}{\partial t^2} + \beta\frac{\partial^3 y}{\partial t \partial x^2} - c^2\frac{\partial^2 y}{\partial x^2} + \varepsilon^4\frac{\partial^4 y}{\partial x^4} = 0 \tag{11}$$

which does not permit classical separation of variables. If we try an exponential solution of the form $e^{st}e^{kx}$, the characteristic equation

$$s^2 + \beta s k^2 - c^2k^2 + \varepsilon^4 k^4 = 0 \approx s^2 + \beta s k^2 - c^2 k^2 \tag{12}$$

$$k = \pm\sqrt{\frac{c^2 - \beta s \pm \sqrt{\left(\beta s - c^2\right)^2 - 4\varepsilon^4 s^2}}{2\varepsilon^4}} \approx \pm\sqrt{\frac{s^2}{c^2 - \beta s}}$$

$$s = \frac{-\beta k^2 \pm \sqrt{\beta^2 k^4 + 4c^2k^2 - 4\varepsilon^4 k^4}}{2} \approx \frac{-\beta k^2 \pm \sqrt{\beta^2 k^4 + 4c^2 k^2}}{2}$$

requires complex numbers ...

# 3 Phasor notation

Recalling the relationships

$$\begin{aligned} e^{\pm i\omega t} &= \cos\omega t \pm i\sin\omega t \\ \cos(\omega t) &= \mathrm{Re}\left(e^{\pm i\omega t}\right) \\ \pm\sin(\omega t) &= \mathrm{Im}\left(e^{\pm i\omega t}\right) \end{aligned}$$

we express phase in a traveling wave; *e.g.,* in the upstream direction, by writing

$$\begin{aligned} y &= \cos(\kappa x + \omega t) \\ &= \cos\kappa x \cos\omega t - \sin\kappa x \sin\omega t \\ &= \cos(\kappa x)\,\mathrm{Re}\left(e^{i\omega t}\right) - \sin(\kappa x)\,\mathrm{Im}\left(e^{i\omega t}\right) \\ &= \cos(\kappa x)\,\mathrm{Re}\left(e^{i\omega t}\right) + \sin(\kappa x)\,\mathrm{Re}\left(ie^{i\omega t}\right) \\ &= \mathrm{Re}\left([\cos\kappa x + i\sin\kappa x]\, e^{i\omega t}\right) \end{aligned}$$

The variables $x$ and $t$ are separated: $[\cos\kappa x + i\sin\kappa x]$ represents the mode shape, with the real-part representing in-phase motion, and the imaginary part the component of the motion which is ninety degrees out-of-phase in time.